

GEOMETRIC AND CONDENSED MATTER INTERPRETATION OF SM FERMIONS AND GAUGE FIELDS

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ABSTRACT. We present the bundle $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, with a geometric Dirac equation on it, as a three-dimensional geometric interpretation of the SM fermions. Each $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ describes two Dirac particles. It has a doubler-free staggered spatial discretization on the lattice space $(\text{Aff}(3) \otimes \mathbb{C})(\mathbb{Z}^3)$. This space has a simple physical interpretation as a phase space of a lattice of cells.

We find the SM $SU(3)_c \times SU(2)_L \times U(1)_Y$ action on $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ to be a maximal gauge action preserving $E(3)$ symmetry, symplectic structure, and anomaly freedom, and which can be constructed using two simple types of gauge-like lattice fields: Wilson gauge fields and correction terms for lattice deformations.

The lattice fermion fields we propose to quantize as low energy states of a canonical quantum theory with \mathbb{Z}_2 -degenerated vacuum state. We construct anticommuting fermion operators for the resulting \mathbb{Z}_2 -valued (spin) field theory.

A metric theory of gravity compatible with this model is presented too.

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1. INTRODUCTION

After the success of relativity, the interest of modern physics has been centered on four-dimensional spacetime. If a concept requires a preferred frame, this is, for many physicists, sufficient to reject it. Of course, to be acceptable, a theory with preferred frame has to explain the observable relativistic symmetry. But this is possible: In [21], GR in harmonic gauge – and, especially, the Einstein equivalence principle – is derived from principles of condensed matter theory. The basic ideas of this derivation we present here in appendix A. Once this basic problem is solved, there seems to be no decisive argument against a preferred frame.

One of the assumptions of this theory is, that matter fields describe material properties of the condensed matter. As a consequence, we need a condensed matter model for the SM fields too. The aim of this paper is to present such a model. We have found a three-dimensional geometric interpretation of SM fermions as $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, together with a doubler-free discretization on the lattice space $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$, which allows a condensed matter interpretation, as the phase space of a lattice of cells. Moreover, this model allows, essentially, to compute the SM gauge group and its action on the fermions. Thus, all SM fields observed until now can be described in this way.

Let's start with the bundle $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, which we propose as a three-dimensional geometric interpretation of the SM fermions. The bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ describes an electroweak doublet.¹ Each of the $3 \cdot (3+1)$ components $(a_\mu^i) \in \text{Aff}(3)$ of an affine transformation we associate with such an electroweak doublet: The upper index i denotes the generation, $\mu = 0$ the leptonic sector, $\mu > 0$ the quark sector, and the three positive values $\mu \in \{1, 2, 3\}$ define the three quark colors.

On the bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ exists a three-dimensional geometric Dirac operator – an analogon of the Dirac-Kähler operator on $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^4)$. This operator is sufficient to define the Dirac matrices α^i . We find also natural operators I_i as well as $\beta = \gamma^0$ on $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. The Dirac equation we define in its original Dirac form $i\partial_t\psi = H\psi$, as an evolution equation on $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. This equation contains eight complex fields and describes a doublet of Dirac particles.

In analogy with the staggered discretization of the four-dimensional bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^4)$, we have also a staggered discretization of the three-dimensional Dirac operator. It lives on a three-dimensional spatial lattice \mathbb{Z}^3 . It is a staggered discretization, with only one complex component on each lattice node, and eight different types of lattice nodes. Similar to the four-dimensional staggered discretization of $\Lambda(\mathbb{R}^4)$ on \mathbb{Z}^4 (see [22],[2]), it is a doubler-free discretization of $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. In other words, we obtain a lattice evolution equation on a three-dimensional lattice $\mathbb{C}(\mathbb{Z}^3)$, which gives, in the continuous limit, two Dirac fermions.

For all SM fermions (the bundle $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$) we obtain a first order lattice equation on $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$. This lattice space has a physical interpretation as the phase space for a three-dimensional lattice of elementary cells, where the state of each cell is described by a single affine transformation from a standard reference cell (see figure 1).

¹Independent of this paper, three-dimensional geometric fermions have been proposed by Daviau [6]. The idea that geometric fermions may be used to describe electroweak doublets has been proposed by Hestenes [13].



FIGURE 1. The space $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$ of the lattice model suggests an interpretation as the phase space (with configuration space $\text{Aff}(3)(\mathbb{Z}^3)$) of a three-dimensional lattice of deformable three-dimensional cells. The configuration of each cell is described by an affine transformation from a standard reference cell.

This physical interpretation gives us two important structures: First, a symplectic structure of the phase space, second, a natural action of the Euclidean group $E(3)$. These structures may be used to restrict the gauge groups. For a compact gauge group we can always construct a preserved Euclidean metric, which, together with a preserved symplectic structure, allows to construct a preserved complex structure. Thus, preservation of the symplectic structure requires unitarity of the gauge groups.

The left action of $E(3)$ on $\text{Aff}(3)(\mathbb{Z}^3)$ transforms the lattice as a whole. The requirement of preserving this symmetry for the gauge groups consists of two parts:

- To commute with rotations, gauge groups have to preserve generations and to act on all three generations in the same way. This holds for all SM gauge fields.
- To commute with translations, one direction in the leptonic sector has to be preserved. All SM gauge fields leave right-handed neutrinos and their antiparticles invariant, thus, a common invariant direction exists in the SM.

Thus, for an appropriate identification of the invariant direction, all SM gauge fields preserve $E(3)$ symmetry.

The lattice theory also leads to another important restriction for the gauge fields: We have to define an appropriate lattice model for the gauge fields. A well-known way to put gauge fields on the lattice are Wilson gauge fields. Their modification to a three-dimensional lattice with continuous time is trivial. But Wilson gauge fields cannot act in a nontrivial way inside the doublets $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, because these are represented on the lattice as $\mathbb{C}(\mathbb{Z}^3)$, which leaves only $U(1)$. Thus, Wilson gauge fields have to have the same charge on all parts of a doublet. The maximal group

of Wilson gauge fields compatible with $E(3)$ symmetry and symplectic structure is $U(3) \cong SU(3)_{QCD} \times U(1)_B$.

We find another modification of the lattice equations which, in the large distance limit, leads to a gauge-like interaction term for fermions. It describes correction terms for lattice deformations. As a consequence, the coefficients depend only on the geometry of the lattice, thus, has to preserve doublets $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, and act identically on all doublets. The maximal group of this type compatible with $E(3)$ symmetry and symplectic structure is generated by chiral $U(2)_L \cong SU(2)_L \times U(1)_L$ and a vector field $U(1)_{\tilde{\gamma}}$ with charge $I_{\tilde{\gamma}} = I_3 - \frac{1}{2}$.

The EM field does not fit into any of the two types. But it can be constructed as a combination of them, by the simple formula $Q = 2I_B + I_{\tilde{\gamma}}$. Thus, our two types of gauge-like lattice fields are already sufficient to construct all SM gauge fields.

Last not least, we have to look at the additional fields – The field $U(1)_B$ with baryon charge I_B , and the diagonal $U(1)_L$ of the weak group $U(2)_L$. Above are, in the presence of the other SM fields, anomal. If we, as a last condition, add anomaly freedom, we can get rid of them too. Thus, we have, essentially, computed the SM gauge group.

Of course, there are a lot of things left to future research. We have not considered yet the mass terms and the Higgs sector. They break $E(3)$ symmetry, thus, to describe them, we need some spontaneous $E(3)$ -symmetry breaking. Once the broken symmetry is $E(3)$, it is not clear if we need a separate Higgs sector at all. This has to be left to future research.

What about quantization? The first problem is fermion quantization. We use classical, commuting c-number fields in the lattice theory $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$, not Grassmann variables as in the Berezin approach to fermion quantization. Thus, we need a completely different quantization scheme for fermions.

We propose such an alternative in section 5. We consider canonical quantization of a field with \mathbb{Z}_2 -symmetric degenerated vacuum state. The lowest energy states of such a field define a \mathbb{Z}_2 -valued (spin) field, yet with commuting operators on different lattice nodes. Then we define anticommuting fermion operators on this space. The transformation is nonlocal and depends on an order between the lattice nodes. We fix such an order and motivate this choice. As a result, the staggered lattice Dirac operator in fermion operators may be obtained from a simpler, non-staggered, symmetric operator in terms of the spin field operators.

A new approach is required for gauge field quantization too. The reason is that the gauge-like lattice fields, which describe lattice deformations, do not have exact gauge invariance on the lattice. This has to be left to future research. Nonetheless, we can already suggest an approach which does not lead to unitarity violations. Quantization of gravity has to follow the scheme of quantization of condensed matter theories. The details have been left to future research too.

2. GEOMETRIC INTERPRETATION OF SM FERMIONS

Let's consider now the geometric interpretation of the SM fermions $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. Throughout this paper, we do not consider mass terms. Without the mass terms, the three generations of SM fermions appear completely identical, simply as three identical copies of the same representation of the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)$.

The group $\text{Aff}(3)$ is the group of three-dimensional affine transformations $y^i = a_j^i x^j + a_0^i$ on \mathbb{R}^3 . Each a_μ^i we can identify with an electroweak doublet of the SM according to the following simple rules: The upper index $i, 1 \leq i \leq 3$ defines the generation. The translational components a_0^i we identify with the leptonic sector. The linear part $a_j^i, j > 0$ we identify with the quark sector. The lower index $j, 1 \leq j \leq 3$ denotes the color of the quark doublet.

This identification of the $3 \times (3+1)$ SM doublets with a $3 \times (3+1)$ affine matrix may be considered, up to now, as pure numerology. But it defines a natural action of the Euclidean group $E(3)$, by multiplication from the left. This action commutes with all gauge fields and plays an important part in the computation of the SM gauge action.

Each electroweak doublet is defined by the bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. It is assumed here that right-handed neutrinos exists, so that neutrinos form usual Dirac particles. Thus, qualitatively there is no difference between electroweak quark doublets and electroweak lepton doublets. Above contain two Dirac particles. The bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ consists of three-dimensional complex inhomogeneous differential forms

$$\begin{aligned}
 (1) \quad \Psi &= \sum_{\kappa_i \in \{0,1\}} \psi_{\kappa_1 \kappa_2 \kappa_3}(x) e^{\kappa_1 \kappa_2 \kappa_3} \\
 &= \psi_{000}(x) + \psi_{100}(x) dx^1 + \psi_{010}(x) dx^2 + \psi_{001}(x) dx^3 \\
 &+ \psi_{110}(x) dx^1 \wedge dx^2 + \psi_{011}(x) dx^2 \wedge dx^3 + \psi_{101}(x) dx^1 \wedge dx^3 \\
 &+ \psi_{111}(x) dx^1 \wedge dx^2 \wedge dx^3.
 \end{aligned}$$

Thus, we have $1+3+3+1 = 8$ complex functions, which gives two Dirac fermions. This allows a physical interpretation in terms of a standard model electroweak doublet. The use of a three-dimensional bundle is essential. In spacetime, we have only the bundle $\mathbb{C} \times \Lambda(\mathbb{R}^4)$, with the Dirac-Kähler equation [16], which describes four Dirac fermions.

On the external bundle $\Lambda(\mathbb{R}^d)$ exists a natural geometric Dirac operator D as a square root of the Laplace operator $\Delta = D^2$. For a general metric, the definition is given in appendix B. In the Euclidean case $g_{\mu\nu} = \delta_{\mu\nu}$, this Dirac operator has the form

$$(2) \quad D = d + d^* = -i\alpha^i \partial_i.$$

with operators α^i which fulfill the anticommutation relations $\{\alpha^i, \alpha^j\} = 2\delta^{ij}$. Now, together with the skew-symmetric α^i , it is useful to consider also corresponding symmetric operators β^i . They may be defined by

$$(3) \quad d - d^* = -i\beta^i \partial_i.$$

Together, they define a set of generators of $M_{2^d}(\mathbb{R}) \cong Cl^{d,d}(\mathbb{R})$:

$$(4) \quad \{\alpha^i, \alpha^j\} = 2\delta^{ij}, \quad \{\alpha^i, \beta^j\} = 0, \quad \{\beta^i, \beta^j\} = -2\delta^{ij}.$$

For $d = 3$, the explicit representation of the matrices α^i, β^i is:

$$\begin{aligned}
-i\alpha^i\partial_i\Psi &= \begin{pmatrix} 0 & -\partial_3 & -\partial_2 & 0 & -\partial_1 & 0 & 0 & 0 \\ +\partial_3 & 0 & 0 & -\partial_2 & 0 & -\partial_1 & 0 & 0 \\ +\partial_2 & 0 & 0 & +\partial_3 & 0 & 0 & -\partial_1 & 0 \\ 0 & +\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & -\partial_1 \\ +\partial_1 & 0 & 0 & 0 & 0 & +\partial_3 & +\partial_2 & 0 \\ 0 & +\partial_1 & 0 & 0 & -\partial_3 & 0 & 0 & +\partial_2 \\ 0 & 0 & +\partial_1 & 0 & -\partial_2 & 0 & 0 & -\partial_3 \\ 0 & 0 & 0 & +\partial_1 & 0 & -\partial_2 & +\partial_3 & 0 \end{pmatrix} \begin{pmatrix} \psi_{000} \\ \psi_{001} \\ \psi_{010} \\ \psi_{011} \\ \psi_{100} \\ \psi_{101} \\ \psi_{110} \\ \psi_{111} \end{pmatrix} \\
-i\beta^i\partial_i\Psi &= \begin{pmatrix} 0 & +\partial_3 & +\partial_2 & 0 & +\partial_1 & 0 & 0 & 0 \\ +\partial_3 & 0 & 0 & +\partial_2 & 0 & +\partial_1 & 0 & 0 \\ +\partial_2 & 0 & 0 & -\partial_3 & 0 & 0 & +\partial_1 & 0 \\ 0 & +\partial_2 & -\partial_3 & 0 & 0 & 0 & 0 & +\partial_1 \\ +\partial_1 & 0 & 0 & 0 & 0 & -\partial_3 & -\partial_2 & 0 \\ 0 & +\partial_1 & 0 & 0 & -\partial_3 & 0 & 0 & -\partial_2 \\ 0 & 0 & +\partial_1 & 0 & -\partial_2 & 0 & 0 & +\partial_3 \\ 0 & 0 & 0 & +\partial_1 & 0 & -\partial_2 & +\partial_3 & 0 \end{pmatrix} \begin{pmatrix} \psi_{000} \\ \psi_{001} \\ \psi_{010} \\ \psi_{011} \\ \psi_{100} \\ \psi_{101} \\ \psi_{110} \\ \psi_{111} \end{pmatrix}
\end{aligned}$$

The last Dirac operator γ^0 can be obtained now as

$$(5) \quad \beta = \gamma^0 = \beta^1\beta^2\beta^3\alpha^1\alpha^2\alpha^3 = \alpha^1\beta^1\alpha^2\beta^2\alpha^3\beta^3,$$

and appears to be a diagonal operator, which measures the \mathbb{Z}_2 -graduation of $\Lambda(\mathbb{R}^3)$. The matrices α^i, β define a representation of the standard Dirac algebra

$$(6) \quad \{\alpha^i, \alpha^j\} = 2\delta^{ij}; \quad \{\alpha^i, \beta\} = 0; \quad (\alpha^i)^2 = \beta^2 = 1.$$

For the (massless) Dirac equation we prefer to use the original form, as proposed by Dirac, with the operators α^i :

$$(7) \quad i\partial_t\Psi = -i\alpha^i\partial_i\Psi.$$

The operators I_i defined by

$$(8) \quad 2i\varepsilon^{ijk}I_i = \beta^j\beta^k,$$

define a vector representation of the isospin algebra $\mathfrak{su}(2)$. We identify them with the (weak) vector isospin $I_i = \tau_L^i + \tau_R^i$. The I_i commute, as they should, with the Dirac equation as well as with γ^0 . Thus, the operator I_3 may be used to split the bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ into two parts with eigenvalues $I_3 = \pm\frac{1}{2}$, so that each of the parts contains a full representation of the Dirac algebra.

An interesting question is how the spinor representation $\sigma^{ij} = \alpha^i\alpha^j$ on the Dirac particles is connected with the representation $\mathfrak{so}(3)$ of geometric rotations of the bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. The answer is that geometric rotations are generated by the operators ω^{ij} defined by

$$(9) \quad \omega^{ij} = \alpha^i\alpha^j - \beta^i\beta^j = \sigma^{ij} - 2i\varepsilon^{ijk}I_i.$$

Thus, the true, geometric rotations of our geometric interpretation are a combination of spinor rotations and isospin rotations.

The operator $\gamma^5 = -i\alpha^3\alpha^2\alpha^1$ turns out to be the (modified) geometric Hodge * operator (83).

2.1. Symplectic structure. We have a complex structure in our geometric interpretation. Now, every complex structure defines a natural symplectic structure $\omega = dz \wedge d\bar{z}$. We know that all the SM gauge groups are unitary groups, thus, they preserve the complex structure. As a consequence, they also preserve the symplectic structure. Therefore, we can postulate the following:

Postulate 1. *All gauge fields preserve the symplectic structure derived from the complex structure of $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$.*

The question we want to consider here is if we really need the complex structure. May be the symplectic structure is already sufficient? Or do we obtain, in this way, some additional gauge fields? No, at least as long as we consider only compact gauge groups. For compact gauge groups, we have the invariant Haar measure, and it has a finite norm. This allows to construct, for a given action of a compact group, an invariant Euclidean norm $\langle \cdot, \cdot \rangle$. All we have to do is to start with an arbitrary norm $\langle \cdot, \cdot \rangle_0$ and to compute the average of the Haar measure:

$$(10) \quad \langle a, b \rangle = \int \langle ga, gb \rangle_0 dg.$$

The resulting Euclidean distance $\langle \cdot, \cdot \rangle$ is already preserved by the gauge group action. Once we have a preserved Euclidean metric together with a preserved symplectic structure, we can already construct a preserved complex structure by the rule

$$(11) \quad \omega(a, ib) = \langle a, b \rangle.$$

As a consequence, our second postulate is sufficient to restrict the gauge group to an unitary group. Thus, in the geometric interpretation we can forget about the complex structure and restrict ourself to the symplectic structure. Thus, we can interpret the space $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ as a phase space.

2.2. Euclidean symmetry. On $\text{Aff}(3)$, we have a well-defined left action of the Euclidean symmetry group $E(3) \subset \text{Aff}(3)$.

The action of the rotation group $O(3) \subset E(3)$ extends immediately to $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ as

$$(12) \quad \omega : \Psi_\mu^i \rightarrow \omega_j^i \Psi_\mu^j.$$

In terms of our interpretation, these rotations rotate the three generations of the SM. Now, all SM gauge groups preserve generations. (Remember that we consider here the massless case, thus, define generations in such a way that they contain electroweak doublets completely.) Moreover, they act on the different generations in exactly the same way. As a consequence, they commute with the action of our group of rotations $O(3)$.

Let's extend now the action of the subgroup of translation $T^3 \subset E(3)$ on $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. For this purpose, we have to define a shift operator

$$(13) \quad t : \Psi_0^i \rightarrow \tau(t^i) \Psi_0^i,$$

where $\tau(t) : \Psi \rightarrow \Psi'$ defines a scalar shift operator on $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. This is an action of \mathbb{R} on $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ and should not depend on x . Therefore, it is uniquely defined by a single shift vector $\vec{c} = (c_\kappa) \in \mathbb{C}^8$ as

$$(14) \quad \vec{c} = \tau(1)\Psi - \Psi,$$

which we name the “direction of translation”. After this, translations are defined as $\tau(t)\psi_\kappa \rightarrow \psi_\kappa + tc_\kappa$ for all κ , and we have extended the definition of translations from $\text{Aff}(3)$ to $(\text{Aff}(3) \otimes \mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$.

In our interpretation, translations act, by shifts, only on the leptonic doublets. Once we already have found that rotations commute with all gauge groups, it would be nice to have a similar property for translations too. So, what does it mean for the gauge groups to commute with translations? The answer is simple — the gauge groups have to leave the translational direction \vec{c} of the leptonic sector invariant. Now, the leptonic sector contains a part which is left invariant by all SM gauge fields — the right-handed neutrinos and their antiparticles. Thus, if we identify the direction of translation \vec{c} in such a way that it is inside the right-handed neutrino sector, then all SM gauge fields preserve translational symmetry too.

Thus, for an appropriate definition of the direction of translation \vec{c} , all SM gauge fields preserve the complete $E(3)$ symmetry. This property of the SM gauge fields we use in the following as a postulate:

Postulate 2. *All gauge fields preserve the $E(3)$ symmetry defined by the left action of $E(3)$ on $\text{Aff}(3)$.*

Note that this observation gives our $E(3)$ symmetry large explanatory power. It explains why all SM gauge fields preserve generations and act in the same way on the three generations. Moreover, it excludes a lot of very interesting natural and symmetric extensions of the SM:

- The extension of $SU(3)_c$ to $SU(4)_c$ with lepton charge as a fourth color, which is part of the Pati-Salam extension of the SM [18],
- the left-right-symmetric extension of $SU(2)_L \times U(1)_Y$ to $U(1)_{B-L} \times SU(2)_L \times SU(2)_R$, which is also part of the Pati-Salam extension of the SM [18],
- and all GUTs which use at least one of these extensions as a subgroup, especially $SO(10)$ GUT.

Indeed, all these extensions of the SM act on right-handed neutrinos in a non-trivial way, and, therefore, do not leave any direction invariant. As a consequence, they cannot commute with any implementation of the translations.

Nonetheless, these principles are not yet sufficient to compute the SM gauge group. There remain interesting nontrivial extensions like $SU(5)$ GUT [10] or chiral color with $SU(3)_L \times SU(3)_R$ instead of $SU(3)_c$ [9].

3. THE LATTICE DIRAC OPERATOR

Let’s consider now a discretization of our Dirac equation in space, leaving time continuous. Using naive central differences, we obtain the following lattice equation:

$$(15) \quad i\partial_t\psi_\kappa(n) = \sum_i -i(\alpha^i)_{\kappa'}^{\kappa} (\psi_{\kappa'}(n+h_i) - \psi_{\kappa'}(n-h_i))$$

on the lattice space $\Omega = \mathbb{C}^8(\mathbb{Z}^3)$.

It is easy to see that this lattice equation contains eight doublers. Indeed, let’s define eight so-called “staggered” sublattices, labelled by $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \{0, 1\}^3$, defined by the condition

$$(16) \quad \Omega^\lambda = \{\psi_\kappa(n) | n = \kappa + \lambda \pmod{2}\}$$

so that $\Omega = \sum_{\lambda} \Omega^{\lambda}$. It is easy to see that the naive lattice Dirac equation preserved the decomposition into the staggered sublattices. As a consequence, it is easy to get rid of the doublers, and sufficient to preserve only one of the eight sublattices Ω^{000} , with $\lambda = \{0, 0, 0\}$. Thus, our staggered sublattice is defined by the condition

$$(17) \quad n = \kappa \pmod{2}.$$

This doubler-free lattice equation (15),(17) can be obtained from a much more general, geometric construction, which is presented in appendix B.1. It is the same geometric construction which gives, in the case of the four-dimensional Dirac-Kähler equation [16] on the spacetime bundle $\Lambda(\mathbb{R}^4)$, the staggered fermions [22] in lattice gauge theory (see [2]).

Now, it is interesting to see what happens with the other operators we have defined in the continuous limit. On Ω , the operators $\alpha^i, \beta^i, I_i, \gamma^5$ and the shift operators $\tau_i : \Psi(n) \rightarrow \Psi(n + h_i)$ are well-defined. Unfortunately, they do not preserve the decomposition into staggered subspaces. Fortunately, there are natural replacements for these operators which already preserve Ω^{000} . For the generators α^i, β^i of $Ct^{d,d}(\mathbb{R})$ we obtain:

$$(18) \quad \tilde{\alpha}^i = \alpha^i \tau_i, \quad \tilde{\beta}^i = \beta^i \tau_i.$$

For the other operators I_i, γ^5 we can use the same formulas we have used in the continuous limit to compute them:

$$(19) \quad \tilde{\gamma}^5 = -i\tilde{\alpha}^3 \tilde{\alpha}^2 \tilde{\alpha}^1 = \gamma^5 \tau_1 \tau_2 \tau_3$$

$$(20) \quad 2i\varepsilon^{ijk} \tilde{I}_i = \tilde{\beta}^j \tilde{\beta}^k = \beta^j \beta^k \tau^j \tau^k$$

Now, the operators $\tilde{\gamma}^5$ and \tilde{I}_i generate an interesting group \mathcal{A} of operators associated with lattice shifts:

Theorem 1. *The group \mathcal{A} of operators generated by $\tilde{\gamma}^5$ and $2\tilde{I}_i$ has the following properties:*

- *It preserves the staggered subspaces Ω^{λ} .*
- *It preserves the lattice Dirac equation.*
- *There exists an epimorphism $\pi : \mathcal{A} \rightarrow \mathbb{Z}^3$ named “underlying shift operator”.*
- *$\text{Ker}\pi \cong \mathbb{Z}_2$ and acts by pointwise multiplication.*

Note one advantage of using the original form $D = \alpha^i \partial_i$ of the Dirac equation here: γ^5 does not anticommute, but commute with the Dirac equation. For a shift operator $\tau \in \mathbb{Z}^3$, the equation $\pi(\tau) = \tau$ defines the operator $\tilde{\tau}$ modulo its sign.

3.1. The cellular lattice model. Let’s forget, for some time, about the staggered character of the lattice Dirac equation. Then, the lattice space of the discretization Ω^{000} is simply $\mathbb{C}(\mathbb{Z}^3)$, with a single complex number on each lattice node. For all SM fermions, we obtain the lattice space $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$.

Note also that we have a first order lattice equation on it. This suggests an interpretation of $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$ as a phase space of some physical system:

$$(21) \quad z_{\mu}^i(n) = a_{\mu}^i(n) + i\pi_{\mu}^i(n).$$

with configuration variables $a_\mu^i(n) : \mathbb{Z}^3 \rightarrow \text{Aff}(3)$ and momentum variables $\pi_\mu^i(n)$. On the phase space $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$ we have the standard symplectic structure

$$(22) \quad \omega = \sum_{i,\mu,n} da_\mu^i(n) \wedge d\pi_\mu^i(n).$$

Then, the configuration space $\text{Aff}(3)(\mathbb{Z}^3)$ appears in a natural way if we have a regular lattice of deformable cells (see figure 1). Here, each cell is described by an affine transformation from some standard reference cell. This reference cell is assumed to be located in the origin.

Now, to have such a simple model is, of course, nice and beautiful. But is it only an otherwise useless toy, or is it helpful to explain the physics of the SM? We want to show here that this model has physical importance.

First, of course, this model gives the symplectic structure, which we have used in section 2.1 to derive the unitarity. Thus, the model allows to explain our postulate 1.

But it seems helpful to explain Euclidean symmetry too. Of course, Euclidean symmetry is not a property of the full SM, the mass matrices break this symmetry. Thus, we need some spontaneous symmetry breaking to explain the SM masses. Nonetheless, the lattice model allows to answer the following simple question: Why do we have to use the left action of $E(3)$ on $\text{Aff}(3)$, instead of the right or adjoint action? For this purpose, let's see what happens with a lattice of deformed cells if we apply the different actions of $E(3)$:

We consider an almost regular lattice. Then we have approximately

$$(23) \quad a_j^i(n) \approx \delta_j^i, \quad a_0^i(n) \approx n_i h,$$

Now, we see that the left action of a rotation rotates the lattice as a whole, including the shifts $a_0^i(n)$. Instead, the right action of a rotation leaves the cells on their places $n_i h$ and rotates them around these places. This, obviously, changes the connection between neighbour cells. Instead, the left action rotates the lattice as a whole, leaving the local geometry unchanged. Thus, the left action seems much more likely to be a symmetry of the theory. In this sense, our cellular model is useful to explain our postulate 2 as well.

But the most important consequence of the cellular lattice model is that we can apply now condensed matter theory. Especially we can, in the large distance limit, define density, velocity, and a stress tensor, and postulate continuity and Euler equations. But this is what we need to incorporate gravity into the model. A metric theory of gravity with GR limit, based on such an "ether concept", has been proposed in [21]. We give a short introduction in appendix A.

4. LATTICE GAUGE FIELDS

While our postulates 1 and 2 impose strong restrictions for the gauge group of the SM, we are yet far away from computing the SM gauge group. There are gauge groups much larger than $SU(3)_c \times SU(2)_L \times U(1)_Y$ compatible with these postulates.

But the consideration of the lattice theory allows to impose another type of restrictions: It should be possible to "put the gauge action on the lattice". We will see that this gives the additional restrictions we need to compute the SM gauge

group almost exactly. The remaining possibilities for additional gauge fields will be killed by the standard condition of anomaly freedom.

4.1. Strong fields as Wilson gauge fields. The classical way to incorporate gauge fields into a lattice theory are Wilson gauge fields. The classical formalism of Wilson gauge fields, even if it was developed for spacetime lattices \mathbb{Z}^4 instead of our lattice of cells \mathbb{Z}^3 , needs only a sufficiently obvious, minor modification. This is caused by the fact that we have no discrete structure in time direction. Formally, it looks like time remaining continuous. This requires a mixed form for the definition of the gauge field: The temporal component $A_0(n, t) \in \mathfrak{g}$ is, like in the continuous case, a function with values in the Lie algebra, but defined on the lattice nodes. Instead, the spatial (vector potential) part A_i is described, as usual for Wilson gauge fields, by Lie group valued functions $U(n, i, t) \in G$ located on the edges $n, n + h_i$ of the lattice. The most important, defining property of the Wilson gauge field remains unchanged too: The lattice gauge symmetry is defined by a gauge-group-valued lattice function $g(\cdot) \in \mathbb{Z}^3 \rightarrow G$ which acts pointwise on the lattice $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$ and is uniquely defined by a gauge action $G \times \mathbb{C} \times \text{Aff}(3) \rightarrow \mathbb{C} \times \text{Aff}(3)$. The gauge transformation acts in the following way:

$$(24) \quad \Psi_\mu^i(n, t) \rightarrow (g(n, t)\Psi)_\mu^i(n, t)$$

$$(25) \quad U(n, i, t) \rightarrow g(n, t)U(n, i)g^{-1}(n + \vec{h}_i, t),$$

$$(26) \quad A_0(n, t) \rightarrow g(n, t)A_0(n, t)g^{-1}(n, t) - (\partial_t g(n, t))g^{-1}(n, t).$$

This definition of the gauge action (26) shows that not all imaginable gauge actions may be defined in this way. Indeed, the gauge action can act only on the generation and color indices. Inside a doublet $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, which is represented on the lattice as $\mathbb{C}(\mathbb{Z}^3)$, it cannot act in a nontrivial way. For a fixed doublet of generation i and color μ , there is only a single complex number $z_\mu^i(n)$ in each lattice node. The only possible Wilson gauge action on the lattice $\mathbb{C}(\mathbb{Z}^3)$ is an action of $U(1)$ with the same charge on all parts of the doublet.

Now, we can compute the maximal possible Wilson gauge action which is compatible with our postulates 1 and 2. It should be an unitary group, which acts on all generations in the same way, preserves the generations, thus, does not act on the generation index i . Then, it acts with the same charge on all parts of electroweak doublets, thus, cannot act on the doublet indices κ . Thus, it can act only on the remaining index μ . This gives $U(4)$ as the maximal gauge group. Moreover, to commute with translations, it has to leave the translational direction \vec{c} in the leptonic sector invariant. But then it has to act trivially on the leptonic sector $\mu = 0$. What remains is the group $U(3)$ acting on the color index $\mu > 0$. Its special subgroup $SU(3)$ is, obviously, the color group $SU(3)_c$ of the SM. The other field is the diagonal $U(1)_B$ with the baryon charge I_B .

Thus, the consideration of Wilson lattice gauge fields has given us an important part of the SM gauge group — the strong interactions.

4.2. Correction terms for lattice deformations. Assume our lattice \mathbb{Z}^3 is not exactly regular but slightly deformed. This requires also a modification of the lattice Dirac equation. What can be said about the general form of the corresponding correction terms?

One way to find a lattice equation for a deformed lattice is to consider the general form of the Dirac equation on a curved background $g_{ij}(x)$, described in appendix

B. Then, we can put the modified continuous equation on the lattice following the scheme presented in appendix B.1. But this much more general lattice equation has, nonetheless, a special structure. The lattice nodes belong to classes of different dimension — points, lines, faces, and cells — and interact only with nodes of neighbour codimension.

For our cellular lattice model, there are now two possibilities. Or the cells of the lattice have some additional properties, like a “dimension”, so that they really interact only with cells of neighbour dimension. In this case, the lattice equation of the deformed lattice would be a discretization of the Dirac equation on $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ with curved background $g_{ij}(x)$. Or there is no such additional structure. In this case, the special structure of the Dirac equation on the regular lattice was only accidental. The general equation for a deformed lattice will destroy some qualitative properties of the continuous Dirac equation. We will find, in the continuous limit, additional terms which do not fit into the form of a Dirac equation on curved background.

On the other hand, however deformed the lattice, the correction coefficients will have geometric nature. They depend on the deformed lattice. We obtain a deformed lattice equation on the same bundle $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$. The deformation of the lattice is certainly no reason to use different lattice equations for the different components $a_\mu^i \in \text{Aff}(3)$. This leads to the following

Postulate 3. *Correction terms for lattice deformations preserve doublets $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$ and act on all doublets in the same way.*

But this is a signature of weak interactions. Thus, we propose the hypothesis that weak interactions describe correction terms for lattice deformations.

Let’s consider now a more general correction term for lattice deformations, one which does not preserve the decomposition of the staggered lattice into the different components ψ_κ . Thus, we consider the lattice $\mathbb{C}(\mathbb{Z}^3)$, with a function $\psi(n)$ on it, and an undistorted lattice Dirac equation. In this equation, we replace now every occurrence of $\psi(n)$ by a sum over values $\psi(n + \xi)$ on neighbour nodes:

$$(27) \quad \psi(n) \rightarrow \psi(n) + \sum_{\xi} g_p^\xi(n) \psi(n + \xi).$$

The geometric coefficients depend on the nodes n themselves as well as on the direction of the neighbour $\xi \in \mathbb{Z}^3$ and on the occurrence p of the term in the undistorted Dirac equation. Using the lattice shift operator $\tau_\xi : \psi(n) \rightarrow \psi(n + \xi)$, we can rewrite the expression as

$$(28) \quad \psi(n) \rightarrow (1 + \sum_{\xi} g_p^\xi(n) \tau_\xi) \psi(n).$$

Now, the lattice shift operator τ_ξ is not a beautiful choice. Especially, for the But, given the fact that it stands together with a complex set of coefficients $g_p^\xi(n)$, we can replace it with different, more beautiful operators, as long as the difference leads only to a redefinition of the $g_p^\xi(n)$. That means, we have the freedom to replace τ_ξ by operators of type $o^\xi(n) \tau_\xi$ with $o^\xi(n) \in \mathbb{C}$. We will use this freedom to replace the shift operators τ_ξ by the operators $\tilde{\tau}_\xi \in \mathcal{A}$ of theorem 1. This gives

$$(29) \quad \psi(n) \rightarrow (1 + \sum_{\xi} \tilde{g}_p^\xi(n) \tilde{\tau}_\xi) \psi(n).$$

This expression can be interpreted as describing interaction terms of ψ with some other field described by various lattice fields $\tilde{g}_p^\xi(n)$, with the operators $\tilde{\tau}_\xi$ as defining the interaction.

Now, we propose here the hypothesis that *the large distance limit of these lattice fields $\tilde{g}_p^\xi(n)$ will be a gauge field $A_i^\xi(x)$* . The proof of this hypothesis has to be left to future research. It would require a more concrete model of the geometric coefficients related with a lattice deformation and, then, a consideration of the large distance limit. But it seems useful, at this point, to remember a major lecture of the Wilson approach to renormalization, namely, large distance universality. The details of the microscopic model may be very different, even qualitatively completely different, nonetheless, the mathematics of the large distance limit may become identical, defining the same universality class. This certainly does not replace a proof of the hypothesis, but, nonetheless, makes it plausible.

On the other hand, we also have to note an essential difference between these correction coefficients for lattice deformations and Wilson gauge fields. Wilson gauge fields have an exact gauge symmetry on the lattice. There is obviously no such thing for these correction coefficients. The consequences of the missing lattice gauge invariance for quantization we will discuss below.

Thus, to name them lattice gauge fields seems misleading and unjustified. Instead, we will name them “gauge-like lattice fields”.

Anyway, even without having a proof of our hypothesis about the classical limit of the lattice fields $\tilde{g}_p^\xi(n)$ as being gauge fields $A_i^\xi(x)$, we know all we need about the interaction terms $\tilde{\tau}_\xi$. They give, in the large distance limit, the eight operators $\{1, \gamma^5, 2I_i, 2I_i\gamma^5\}$. Thus, they give all the operators we need to define weak gauge fields. Thus, if our hypothesis is correct, we have found a way to put the weak gauge group $SU(2)_L$ on the lattice.

Let’s now look at this possibility to construct gauge fields from the other side. What would be the maximal gauge group, which can be obtained in this way?

We have already found, that this gauge group has to preserve doublets and to act on all doublets in the the same way. This is, obviously, in correspondence with the description of the gauge group as generated by linear combinations of the operators $\{1, \gamma^5, 2I_i, 2I_i\gamma^5\}$. Indeed, all of them preserve doublets and act on them in the same way. Preservation of the symplectic structure and rotational symmetry do not give any additional restrictions. But translational symmetry gives such an additional restriction: On the leptonic sector, there has to be a preserved direction.

The maximal gauge group $U(2)_L \times U(2)_R$ generated by the generators $\{1, \gamma^5, 2I_i, 2I_i\gamma^5\}$ does not have any invariant direction. Thus, any group which fulfills our conditions has to be smaller than this group. The preferred direction may be left-handed, right-handed, or a linear combination of above. The last case gives a more rigorous restriction of the group, because as the left-handed part, as the right-handed part would have to be preserved. Thus, we can ignore the last case. Without restriction of generality we, therefore, assume, that the preserved translational direction is right-handed. This gives the group $U(2)_L$ as preserving this direction.

Now, let’s see which part of $U(2)_R$ preserves one translational direction. This will be a subgroup $U(1) \subset U(2)_R$ with charge 0 on the translational direction and charge ± 1 on the orthogonal direction. Without restriction of generality, we can

use here $(1 + \gamma^5)/2(I_3 - 1/2)$ as its charge. This choice locates the translational direction inside the right-handed neutrino sector. Instead of the right-handed charge $\frac{1+\gamma^5}{2}(I_3 - \frac{1}{2})$, we can as well use the corresponding vector charge $I_{\bar{\gamma}} = I_3 - \frac{1}{2}$. Thus, we obtain the following

Theorem 2. *The maximal gauge group, obtained from correction terms for lattice deformations, and compatible with $E(3)$ symmetry and symplectic structure, is the group generated by the chiral $U(2)_L$ and the vector $U(1)_{\bar{\gamma}}$ with charge $I_{\bar{\gamma}} = I_3 - \frac{1}{2}$.*

4.3. The EM field and anomaly freedom. At a first look, we see the EM field does not fit into any of the two classes of gauge-like lattice fields. It acts nontrivially inside doublets, thus, is not a Wilson field. On the other hand, it has different charges on leptons and quarks, thus, does not fit into the weak fields too.

Nonetheless, we have already obtained it. It appears as a linear combination of a strong field and a weak field. Indeed, we have

$$(30) \quad U(1)_{em} \cong S(U(1)_B \times U(1)_{\bar{\gamma}})$$

with the charge

$$(31) \quad Q = 2I_B + I_{\bar{\gamma}}$$

Thus, we have obtained now all SM gauge fields, and the two types of lattice gauge fields are already sufficient. Thus, we can add the next postulate for the gauge fields:

Postulate 4. *All gauge fields have to be constructed using the following two types of gauge-like lattice fields:*

- *Wilson gauge fields,*
- *correction terms for lattice deformations.*

With this postulate, we are already very close to the computation of the gauge group of the SM. Indeed, the maximal gauge group which can be obtained with our two types of gauge fields is the group

$$(32) \quad G_0 \cong U(3)_c \times U(2)_L \times U(1)_Y \supset SU(3)_c \times SU(2)_L \times U(1)_Y$$

The difference between the maximal possible gauge group G_0 and the SM gauge group is minimal.

To get rid of the remaining diagonal fields, it is sufficient to remember about the gauge anomaly. The additional diagonal fields are anomalous. Thus, it remains to add

Postulate 5. *The gauge fields have to be anomaly-free.*

and we have finished the computation of the SM gauge group. We have the following theorem:

Theorem 3. *The SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ defines a maximal gauge action on our lattice model $(\mathbb{C} \otimes \text{Aff}(3))(\mathbb{Z}^3)$ which preserves $E(3)$ symmetry and the symplectic structure, gives anomaly freedom, and can be constructed using only Wilson gauge fields and correction terms for lattice deformations.*

Of course, the result of the computation is not unique. There are other maximal gauge groups which fulfill these postulates. For example, we could start with $U(1)_B$ or $U(1)_{\bar{\gamma}}$ and extend them as much as possible. As well, we could use $SU(2)_R$

instead of $SU(2)_L$. It seems reasonable to hope that future research, especially the consideration of the renormalization equations, allows to obtain even better results.

5. FERMION QUANTIZATION

Following Berezin [3], the classical limit of fermion fields are Grassmann-valued fields. This is, obviously, incompatible with our geometric interpretation of fermion doublets as $(\mathbb{C} \otimes \Lambda)(\mathbb{R}^3)$, nor with the lattice model $\mathbb{C}(\mathbb{Z}^3)$, which are classical, commuting, fields, with a standard symplectic structure. The appropriate way to quantize them would be canonical quantization.

Here we present a way to obtain anticommuting fermion fields via canonical quantization. It consists of two parts, with a canonically quantized \mathbb{Z}_2 -valued field (spin field) as the intermediate step. To obtain a \mathbb{Z}_2 -valued field from an \mathbb{R} -valued field, all we need is a \mathbb{Z}_2 -degenerated potential $V(\varphi)$. The lowest energy states, then, already define a \mathbb{Z}_2 -valued field theory. This potential $V(\varphi)$ already requires the breaking of $E(3)$ symmetry, at least if φ is the direction of translation \vec{c} . But this is necessary, because we cannot define a translation on \mathbb{Z}_2 -valued fields, moreover, on fermion fields.

The more non-trivial step is from spin fields to fermion fields, or from commuting to anticommuting operators at different nodes. The lattice operator algebras appear to be isomorph, but the isomorphism is, first, nonlocal, and, second, not natural, depends on some order between different lattice nodes. This leads to a nontrivial transformation and approximation of the Hamilton operator

We consider here canonical quantization of lattice theories with configuration space $Q = \mathbb{R}(\mathbb{Z}^3)$ resp. $Q = \mathbb{Z}_2(\mathbb{Z}^3)$. Our considerations here do not depend on the dimension $d = 3$, so we consider here the more general case $Q = \mathbb{R}(\mathbb{Z}^d)$ resp. $Q = \mathbb{Z}_2(\mathbb{Z}^d)$. Canonical quantization consists of the definition of operators on the Hilbert space $\mathcal{L}^2(Q)$, and a Schrödinger equation

$$(33) \quad i\partial_t \Psi(q, t) = H\Psi(q, t), \quad q \in Q = \mathcal{F}(\mathbb{Z}^3, Y), \quad t \in \mathbb{R}.$$

Thus, we always have continuous time. Note that in the condensed matter interpretation the lattice \mathbb{Z}^3 is not a “discretization of space” \mathbb{R}^3 itself. Instead, it enumerates elementary cells located in a continuous \mathbb{R}^3 , where the state of the cells is described by some affine transformation $\text{Aff}(3) \subset \mathbb{R}^{12}$ of \mathbb{R}^3 . Nonetheless, for the purpose of this section, the lattice \mathbb{Z}^3 may be considered like a “discretization of space” $\mathbb{Z}^3 \subset \mathbb{R}^3$, and our geometric interpretation of \mathbb{R}^{12} plays no role here.

5.1. From fermion fields to spin fields. Spin fields have the configuration space $\mathbb{Z}_2(\mathbb{Z}^d)$. On each lattice node $n \in \mathbb{Z}^d$ we have the Pauli matrices σ_n^i as operators:

$$(34) \quad \sigma_n^i \sigma_n^j = \delta_{ij} + i\varepsilon_{ijk} \sigma_n^k.$$

Spin field operators on different nodes commute:

$$(35) \quad [\sigma_m^i, \sigma_n^j] = 2i\delta_{mn}\varepsilon_{ijk}\sigma_n^k.$$

Instead, following Berezin[3], the fermion field operators ψ_n, ψ_n^* are usually considered to be of qualitatively different nature. They do not fit into the canonical scheme. Especially there is no configuration space Q . Indeed, operators related to different nodes do not commute. Instead, they anticommute:

$$(36) \quad \{\psi_m, \psi_n^*\} = \delta_{mn}, \quad \{\psi_m^*, \psi_n^*\} = \{\psi_m, \psi_n\} = 0.$$

This difference seems to forbid any identification of fermions with spin fields.

Despite this, the two operator algebras appear to be isomorph. The isomorphism is well-known in the theory of Clifford algebras and allows to establish the isomorphism

$$(37) \quad Cl^{N,N}(\mathbb{R}) \cong M_2(Cl^{N-1,N-1}(\mathbb{R})) \cong M_{2^N}(\mathbb{R}).$$

To see this, let's at first transform the operator algebras in each node into an equivalent form, by defining operators ψ_n^i :

$$(38) \quad \psi_n^1 = \psi_n + \psi_n^*, \quad \psi_n^2 = -i(\psi_n - \psi_n^*), \quad \psi_n^3 = -i\psi_n^1\psi_n^2.$$

This gives

$$(39) \quad \psi_n^i\psi_n^j = \delta_{ij} + i\varepsilon_{ijk}\psi_n^k,$$

similar to (34). In these variables, the operators ψ_n^1 and $i\psi_n^2$ generate (for a finite lattice with N nodes) the Clifford algebra $Cl^{N,N}(\mathbb{R})$. On the other hand, $M_{2^N}(\mathbb{R})$ is the operator algebra on the 2^N -dimensional space of \mathbb{Z}_2 -valued functions on the same lattice.

But the isomorphism between these two operator algebras is not natural. It depends on the choice of some order $>$ between the lattice nodes. For a given order, the isomorphism is defined by:

$$(40) \quad \psi_n^{1/2} = \sigma_n^{1/2} \prod_{m>n} \sigma_m^3, \quad \psi_n^3 = \sigma_n^3,$$

$$(41) \quad \sigma_n^{1/2} = \psi_n^{1/2} \prod_{m>n} \psi_m^3, \quad \sigma_n^3 = \psi_n^3.$$

Note also that (different from the σ_n^i) the operators ψ_n^i do not act as local operators on the lattice. Instead, they act like σ_m^3 on other nodes $m > n$. This is a necessary property of such an isomorphism, because, obviously, any local combination of the commuting local operators σ_n^i leads only to another set of commuting local operators.

As a consequence, a Hamilton operator which “looks local” in terms of the ψ_n^i may appear nonlocal in terms of the σ_n^i (which we consider to be “truly local” operators) and reverse. Fortunately, there are important examples of operators where this does not happen. First, there is the operator

$$(42) \quad H_0 = -\frac{1}{2} \sum_n \sigma_n^3 = \frac{1}{2} \sum_n \psi_n^* \psi_n - \psi_n \psi_n^*$$

Let's consider now operators with interactions between neighbour nodes. We are (for reasons which become obvious later) especially interested in the following linear combination:

$$(43) \quad H_D = \frac{1}{2} \sum_{n,i} \sigma_n^1 \sigma_{n+h_i}^1 - \sigma_n^2 \sigma_{n+h_i}^2$$

where h_i are the d basic lattice shifts in the d -dimensional lattice \mathbb{Z}^d .

Now, in the one-dimensional case, we have a natural (up to the sign) order $>$. For this order, we obtain:

$$(44) \quad H_D^{(1)} = i\frac{1}{2} \sum_n (\psi_n^1 \psi_{n+1}^2 + \psi_n^2 \psi_{n+1}^1) = \sum_n \psi_n \psi_{n+1} - \psi_n^* \psi_{n+1}^*$$

Note that our operator is symmetric for spatial inversion $n \rightarrow -n$, but the representation in the asymmetric (in terms of the σ_n^i) operators ψ_n^i hides this symmetry.

5.2. The case of higher dimensions. Unfortunately, the transformation of the Hamilton operator in higher dimensions is not that simple. What we can obtain is only an approximation

$$(45) \quad H_D^{(d)} \approx \tilde{H}_D^{(d)} = \sum_{n,i} \alpha_{n+h_i}^n (\psi_n \psi_{n+h_i} - \psi_n^* \psi_{n+h_i}^*)$$

where

$$(46) \quad \alpha_{n+h_i}^n = \begin{cases} 1 & \text{if } n < n + h_i \\ -1 & \text{else} \end{cases}$$

The accuracy of this approximation obviously depends on the order $>$. Indeed, the error

$$(47) \quad \sigma_n^1 \sigma_{n'}^1 \approx \psi_n^1 \psi_{n'}^2 = \sigma_n^1 \sigma_{n'}^1 \prod_{n < m < n'} \sigma_m^3,$$

resp. for $\sigma_n^2 \sigma_{n'}^2$, depends on the number and location of the nodes m located “between” (according to the chosen ordering) the “neighbour” (according to the lattice \mathbb{Z}^d) nodes n, n' . Now, instead of the simple lexicographic order (which gives $\alpha_{n+h_i}^n = 1$) we propose to use another, more sophisticated order we name “alternating lexicographic order”.

It has to be acknowledged that this order has been designed to give the result below. Fortunately, we can justify this choice of an order in another way: It gives a better approximation of the original Hamiltonian operator, in the sense, that some algebraic properties of the original terms may be preserved exactly.

Note that our interaction terms can be represented as a function of the differences of the operator σ_n^1 and its shift:

$$(48) \quad \sigma_n^1 \sigma_{n+h_i}^1 = 1 - \frac{1}{2} ((1 - \tau_i) \sigma_n^1)^2,$$

where τ_i is the shift operator on the lattice. This follows from $(\sigma_n^1)^2 = 1$ and the commutation relation $[\sigma_n^1, \tau_i \sigma_n^1] = 0$. Now, we propose to use an order which allows to preserve these properties exactly. That means, we want to replace the σ_n^1 by some $\tilde{\sigma}_n^1$ with exactly the same properties:

$$(49) \quad (\tilde{\sigma}_n^1)^2 = 1, \quad [\tilde{\sigma}_n^1, \tau_i \tilde{\sigma}_n^1] = 0,$$

so that

$$(50) \quad \sigma_n^1 \sigma_{n+h_i}^1 \approx \tilde{\sigma}_n^1 \tilde{\sigma}_{n+h_i}^1 = 1 - \frac{1}{2} ((1 - \tau_i) \tilde{\sigma}_n^1)^2.$$

For the simple lexicographic order, we have no way to define such $\tilde{\sigma}_n^1$. But it is possible for the alternating lexicographic order. We define it by induction. Let $>_k$ be the order defined for a k -dimensional lattice \mathbb{Z}^k , and π_k the projection on this lattice defined by the first k coordinates. Then we define $>_{k+1}$ by the following properties:

- if $n_{k+1} \leq m_{k+1}$ then $n \leq_{k+1} m$;
- else if $n_{k+1} (= m_{k+1})$ is even and $\pi_k n \leq_k \pi_k m$ then $n \leq_{k+1} m$;
- else if $n_{k+1} (= m_{k+1})$ is odd and $\pi_k n \leq_k \pi_k m$ then $n \geq_{k+1} m$.

Thus, we use the inverse order inside the odd planes. Now the interaction term can be splitted in the following way:

$$(51) \quad \sigma_n^{1/2} \sigma_{n+h_i}^{1/2} \prod_{n < m < n+h_i} \sigma_m^3 = \tilde{\sigma}_n^{1/2} \tilde{\sigma}_{n+h_i}^{1/2}$$

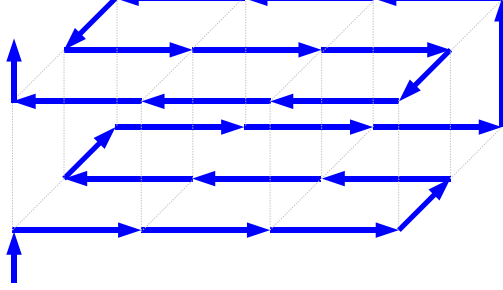


FIGURE 2. The alternating lexicographic order

with

$$(52) \quad \tilde{\sigma}_n^{1/2} = \sigma_n^{1/2} \prod_{\substack{n < m \\ m_i = n_i}} \sigma_m^3, \quad \tilde{\sigma}_{n+h_i}^{1/2} = \sigma_{n+h_i}^{1/2} \prod_{\substack{m < n+h_i \\ m_i = n_i+1}} \sigma_m^3$$

we obtain the properties (49). The key is that for each node m with $n < m$, $m_i = n_i$ the shifted point $m' = \tau_i m$ fulfils $m' < n + h_i$, thus, for each σ_m^3 in the first term we find a corresponding $\sigma_{m'}^3$ in the second term.

For our choice of $>$, the coefficients α_n^n fulfill the following relations:

$$(53) \quad \alpha_m^n = \alpha_{m+2h_i}^{n+2h_i}; \quad \alpha_{n+h_i}^n \alpha_{n+2h_i}^{n+h_i} = 1; \quad \alpha_{n+h_i}^n \alpha_{n+h_i+h_j}^{n+h_i} = -\alpha_{n+h_j}^n \alpha_{n+h_i+h_j}^{n+h_j}.$$

5.3. Transformation of the lattice Dirac operator into staggered form.

Now, the operator $H = \tilde{H}_D + mH_0$ appears to be a lattice Dirac operator. Indeed, let's consider the evolution equation defined by H :

$$(54) \quad i\partial_t \psi_n = [H, \psi_n] = \sum_i \alpha_{n+h_i}^n (\psi_{n+h_i}^* - \psi_{n-h_i}^*) - m\psi_n,$$

$$(55) \quad i\partial_t \psi_n^* = [H, \psi_n^*] = -\sum_i \alpha_{n+h_i}^n (\psi_{n+h_i} - \psi_{n-h_i}) + m\psi_n^*.$$

As a consequence of the relations (53), the evolution equations (54),(55) give

$$(56) \quad \partial_t^2 \psi_n = \sum_i (\psi_{n+2h_i} - 2\psi_n + \psi_{n-2h_i}) - m^2 \psi_n = ((\Delta_{2h} + m^2)\psi)_n,$$

where Δ_{2h} is the lattice Laplace operator with doubled distance $2h_i$ — a Laplace operator on a coarse lattice.

The lattice Laplace operator Δ_{2h} acts independently on 2^d different sublattices. Let's distinguish these sublattices by introduction of 2^d different lattice functions

enumerated by elements of $\kappa = (\kappa_1, \dots, \kappa_d) \in \{0, 1\}^d$. Using the denotation $*\psi_n = \psi_n^*$, we define

$$(57) \quad \psi_\kappa(n) = *\kappa_1 + \dots + \kappa_d \psi_n \quad \text{on} \quad n \equiv \kappa \pmod{2}.$$

Each of the 2^d lattice functions $\psi_\kappa(n)$ is defined on a ‘‘coarse lattice’’ containing the nodes of type $n_i = 2\tilde{n}_i + \kappa_i$ and lattice spacing $2h_i$. Now, the lattice operator Δ_{2h} acts as the simple Laplace operator on each of the 2^d functions $\psi_\kappa(n)$. In the continuous limit, each $\psi_\kappa(n)$ gives a function $\psi^\kappa(x)$ which fulfills the Klein-Gordon equation

$$(58) \quad \partial_i^2 \psi_\kappa(x, t) = \left(\sum_i \partial_i^2 - m^2 \right) \psi_\kappa(x, t) = 0.$$

The lattice Dirac equations (54),(55) now establish a connection between these 2^d lattice fields. We can define now $2^d \times 2^d$ matrices $(\alpha^i)_\kappa^{\kappa'}, \beta_\kappa^{\kappa'}$ so that the original lattice equations (54),(55) transform into

$$(59) \quad i\partial_i \psi_\kappa(n) = [H, \psi_\kappa(n)] = \sum_i -i(\alpha^i)_\kappa^{\kappa'} (\psi_{\kappa'}(n+h_i) - \psi_{\kappa'}(n-h_i)) + m\beta_\kappa^{\kappa'} \psi_{\kappa'}(n)$$

on $n = \kappa \pmod{2}$. Because of the factor $*\kappa_1 + \dots + \kappa_d$ in (57), equation (15) connects only the fields $\psi_\kappa(n)$, and its adjoint only the $(\psi_\kappa(n))^*$.

This equation is our lattice Dirac equation (15) on the staggered lattice (17), but already in its quantized form, with anticommuting fermion operators $\psi_\kappa(n)$.

5.4. From spin fields to scalar fields. Spin fields are already a much more classical object in comparison with the original fermion fields. But we need even more classical objects, namely real-valued fields.

But this is not problematic at all. We can embed the spin field as an effective description of the lowest energy states of a scalar field with a \mathbb{Z}_2 -symmetric potential with two different vacuum states. For example, we can consider φ^4 theory in \mathbb{R}^d with negative mass parameter μ^2 :

$$(60) \quad \mathcal{L} = \frac{1}{2}((\partial_t \varphi)^2 - (\partial_i \varphi)^2) - V(\varphi) \quad \text{with} \quad V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4.$$

The two minima of the potential are $\varphi(x) = \pm\varphi_0$ with $\varphi_0 = \sqrt{\frac{6\mu^2}{\lambda}}$.

If the system is near φ_0 , it is convenient to use the σ -variable $\sigma(x) = \varphi(x) - \varphi_0$ so that

$$(61) \quad V(\sigma) = \frac{1}{2}(2\mu^2)\sigma^2 + \sqrt{\frac{\lambda}{6}}\mu\sigma^3 + \frac{\lambda}{4!}\sigma^4.$$

This describes a scalar field with mass $\sqrt{2}\mu$ and some interactions.

Instead, we are interested only in the lowest energy states of this theory. Let's consider at first the simple case of dimension $d = 0$, where QFT reduces to ordinary quantum theory. If we have energies much below μ , only the two vacuum states $\Psi_\pm(\varphi)$ with $\langle \Psi_\pm | \varphi | \Psi_\pm \rangle \approx \pm\varphi_0$ are important. But the true eigenstates of energy are

$$(62) \quad \Psi_{0/1}(\varphi) = \frac{1}{\sqrt{2}}(\Psi_+(\varphi) \pm \Psi_-(\varphi)).$$

Between them, we have an energy gap of order

$$(63) \quad \Delta = E_1 - E_0 \sim \exp\left(-\int_0^{\varphi_0} \sqrt{V(\varphi) - E_0} d\varphi\right) \sim \exp\left(-\frac{\mu^3}{\lambda}\right).$$

With increasing μ the mass of the σ field increases, but the energy gap Δ decreases exponentially. Without any conspiracy, this leads to two different domains: a high energy domain, with energies of order μ , where the tunneling may be ignored, and a low energy domain, with energies of order Δ , where the whole theory reduces to the two-dimensional space spanned by $\Psi_{0/1}$. Reduction to this subspace gives

$$\begin{aligned} \varphi &\rightarrow \varphi_0 \sigma^1 && \text{with } \varphi_0 = \int \bar{\Psi}_0 \cdot \varphi \Psi_1 d\varphi \\ \pi = \frac{\delta L}{\delta \dot{\varphi}} &\rightarrow \pi_0 \sigma^2 && \text{with } \pi_0 = \int \bar{\Psi}_0 \cdot \partial_\varphi \Psi_1 d\varphi \\ H &\rightarrow H_0 - \frac{1}{2} \Delta \sigma^3 && \text{with } H_0 = \frac{E_0 + E_1}{2}. \end{aligned}$$

For dimension $d > 0$, at least as long as the momentum k is sufficiently small, we have a similar situation for each of the modes $\varphi(x) = \exp(ikx)\varphi$. For sufficiently large μ , and sufficiently low energies under consideration, the theory reduces to an effective theory where we have only two degrees of freedom for each mode. Effectively, the configuration space reduces from $\mathcal{F}(\mathbb{Z}^d, \mathbb{R})$ to $\mathcal{F}(\mathbb{Z}^d, \mathbb{Z}_2)$.

Last not least, let's consider typical lattice theory interaction terms which may appear in the reduction for a Lagrangian of type (60). We consider lattice approximations where only neighbour nodes have nontrivial interaction terms. Let $n, n' = n + h_i$ be these neighbour nodes, $d = 1$. One possibility is to use $\frac{1}{2}(\pi_n + \pi_{n'})$ to approximate $\pi(x)$ and $\frac{1}{h}(\varphi_n - \varphi_{n'})$ to approximate $\partial_i \varphi(x)$. Then, the reduction gives an effective Hamiltonian

$$(64) \quad H = \frac{1}{2}(\pi^2 + c^2(\partial_x \varphi)^2 + V(\varphi)) \rightarrow c_0 + c_1 \sigma_n^1 \sigma_{n'}^1 + c_2 \sigma_n^2 \sigma_{n'}^2 + c_3 \sigma_n^3$$

for some constants c_i . The lattice Dirac operator corresponds to $c_1 = -c_2 = 1$, thus, can be obtained in this scheme.

As a consequence of this quantization method for fermions, we obtain some analogon of a ‘‘supersymmetric partner’’ of the fermions. This partner can be very heavy without any conspiracy. At the current state of research, no indications about their masses can be given.

5.5. Generalization of Bohmian mechanics and Nelsonian stochastics.

A consequence of our approach to fermion quantization is that we obtain, in this way, a new route for the generalization of hidden variable theories like Bohmian mechanics [5] or Nelsonian stochastics [17]. Indeed, these theories have been defined for multi-particle Schrödinger theory. Now, our model, and, even more, the quantization scheme we have used here, is already very close to classical multi-particle Schrödinger theory.

Indeed, we can identify the state $a_\mu^i(n)$ of the cell n with the position of four of its points $a_\mu(n)$. In this sense, the configuration space already coincides with the configuration space of a classical multi-particle theory. Thus, we have already the same configuration space as in Bohmian mechanics as well as in Nelsonian stochastics.

Moreover, we have also used classical canonical quantization. Thus, all the quantum operators are also the same as in Bohmian mechanics or Nelsonian stochastics.

The only difference is that the Schrödinger operator is not exactly of the same form as in multi-particle Schrödinger theory. But the difference is not very big. Indeed, the Hamilton operator (64) is already of the form

$$(65) \quad H = Q(p_i) + V(Q)$$

with some non-degenerated quadratic form $Q(p_i) \geq 0$ of the momentum variables and some potential $V(Q)$ on the configuration space.

Thus, one way to obtain a generalization of Bohmian mechanics and Nelsonian stochastics is to diagonalize the quadratic form $Q(p_i)$ so that, for some new, diagonalized, momentum variables $p'_i(p_j)$, we have

$$(66) \quad Q(p_i) = \sum_j \frac{p'_j}{2m_j}$$

and we have recovered the standard form of multi-particle Schrödinger theory, so that we can use the standard versions of Bohmian mechanics and Nelsonian stochastics.

6. ABOUT GAUGE FIELD QUANTIZATION

A first objection against our construction of weak gauge fields in section 4.2 is that it presents a lattice regularization for chiral gauge field theory. But to obtain such a regularization is a famous problem of chiral lattice gauge theory [12], and there are various no-go theorems for such regularizations.

But the regularization problem of chiral gauge theory is the problem to find a *gauge-invariant* regularization. Our regularization has no exact gauge invariance on the lattice. Instead, we have only approximate gauge invariance, modulo even lattice shifts. Thus, our regularization is not in contradiction with the various no-go theorems for regularizations with exact gauge invariance.

This answer leads, in a natural way, to a second objection. Last not least, people have tried to find regularization with exact gauge invariance not just for fun, but for a good reason — to quantize chiral gauge fields. The problem is that the standard procedure to quantize gauge fields — BRST quantization — depends essentially on exact gauge invariance of the theory. Without exact gauge invariance, it fails miserably. What remains is a non-unitary theory.

But this failure is a special problem of the manifestly Lorentz-covariant Gupta-Bleuer approach to gauge field quantization, which starts with an indefinite Hilbert space structure. Following Gupta [11] and Bleuer [4], in the BRST approach, manifest relativistic invariance is reached using an unphysical “big space” with indefinite Hilbert metric. A physical interpretation of this big Hilbert space would lead to negative probabilities, which is nonsensical. To get rid of the states with negative probability, restriction to an invariant subspace and factorization is used. But these operations depend on exact gauge invariance. If gauge invariance fails, the result is fatal for the whole approach.

But there is an alternative — the earlier approach of Fermi [8] and Dirac [7], where the Hilbert space is definite, but Lorentz covariance is not explicit. Whatever may go wrong, the Hilbert space remains definite, and at least a probability interpretation of the results is possible.

In our approach, weak gauge fields appear, in the large distance limit, as effective fields.² The fundamental theory is not obliged to contain them explicitly. As far as we have developed it until now, is a well-defined and unitary quantum theory, a variant of multi-particle Schrödinger theory. Whatever the complications connected with symmetry breaking, we would not switch to an indefinite Hilbert space, because the main advantage of this choice — manifest Lorentz invariance — cannot be reached in our approach anyway.

This answer leads to the next objection: The manifestly Lorentz-covariant approach has not been introduced without reason too. Manifest Lorentz covariance, on one hand, simplifies computations. This is, obviously, not a decisive argument. More serious is that, in our approach, we do not have relativistic invariance. Indeed, our construction from the start violates Lorentz covariance, and in many different ways: First, we handle time and space in different ways, having a lattice only in space. Then, even a spacetime lattice would violate the symmetry of the continuous limit. Moreover, the operators σ_{ij} , for spatial spinor rotations on our staggered lattice, are nonlocal, and, therefore, do not define an exact representation of the algebra $\mathfrak{su}(2)$. An approach, which violates Lorentz invariance on the fundamental level, has to explain, how it will be recovered in the large distance limit.

Fortunately, this question has been, at least partially, addressed by the derivation of the Einstein equivalence principle (which includes local Lorentz covariance) in our theory of gravity A. Because of the importance of this question we give an introduction into this theory in appendix A.

7. DISCUSSION

Many questions have to be left to future research. This includes:

- Symmetry breaking;
- The search for a Hamilton operator for a general configuration of cells, which would allow the derivation for other regular crystallographic lattices as well as for lattices with deformations and defects;
- The large distance limit, especially renormalization group equations;
- The connection between the SM lattice model and the theory of gravity, which are metaphysically compatible, but mathematically yet unrelated theories; In the large distance limit of our cellular lattice, we will obviously have notions like density ρ , average velocity v^i and some stress tensor σ^{ij} .

Especially symmetry breaking promises to be interesting. First, we need it. The $E(3)$ action does not define a symmetry of the SM. The SM mass terms clearly violate the rotational symmetry between the three generations. Moreover, the whole construction of section 5, which creates effective \mathbb{Z}_2 -fields from the original \mathbb{R} -fields, violates translational symmetry: We cannot add constants to \mathbb{Z}_2 -valued fields. Thus, to obtain the Hamiltonian of the SM, even to obtain fermion fields at all, we have to break $E(3)$ symmetry. Euclidean symmetry is also broken by the EM field, which prefers the direction associated with I_3 , which is also associated with a direction in space.

²Note that, instead, Wilson gauge fields are located on the links between lattice nodes, their related degrees of freedom are not part of the basic, fermionic theory. Instead, the weak gauge fields, described by geometric coefficients of the lattice itself, are already fixed if the lattice itself is fixed. Thus, the related theory appears as an effective field theory.

On the other hand, some arguments for the standard SM theory of symmetry breaking seem to fail. Especially, it is not clear, if we need some Higgs sector for symmetry breaking. One major argument — that spatial symmetry is not broken — fails in our geometric interpretation of the SM fields. Then, we do not need a fundamental theory, with unbroken gauge symmetry, for electroweak gauge fields. Our theory has no lattice gauge symmetry for the weak gauge fields from the start. Note that Wilson gauge fields, with their exact lattice gauge symmetry, correspond to massless gluons, while the weak, chiral fields, which do not allow exact lattice gauge invariance, appear to be massive. Thus, we don't need symmetry breaking to break an exact fundamental gauge symmetry.

Thus, we need symmetry breaking, but for very different reasons, and a different symmetry. Thus, the symmetry breaking may be expected to be very different from the SM symmetry breaking approach.

Despite these open questions, our simple cellular lattice model already allows to describe kinematically all SM particles observed so far, and it is compatible with a metric theory of gravity with GR limit.

APPENDIX A. GRAVITY

For metric theories of gravity there is a simple way to obtain a condensed matter interpretation, closely related to the ADM decomposition [1] or the geometrodynamical interpretation [23]. The preferred frame defines an ADM decomposition of the four-metric $g_{\alpha\beta}$ into a scalar field, a three-vector and a definite three-metric. We identify these fields with density ρ , velocity v^i and stress tensor σ^{ij} of some form of condensed matter in the following way:

$$\begin{aligned}
 (67) \quad g^{00}\sqrt{-g} &= \rho, \\
 g^{0i}\sqrt{-g} &= \rho v^i, \\
 g^{ij}\sqrt{-g} &= \rho v^i v^j - \sigma^{ij}.
 \end{aligned}$$

For these condensed matter fields, we would like to have continuity and Euler equations:

$$(68) \quad \partial_t \rho + \partial_i(\rho v^i) = 0$$

$$(69) \quad \partial_t(\rho v^i) + \partial_i(\rho v^i v^j - \sigma^{ij}) = 0.$$

They coincide with the harmonic conditions for the metric (67):

$$(70) \quad \partial_\alpha(g^{\alpha\beta}\sqrt{-g}) = 0.$$

This condensed matter interpretation is, therefore, possible for all metric theories of gravity, which include the harmonic condition as a physical equation. A simple theory with this property is general relativity in harmonic gauge. One variant of GR in harmonic gauge is to add a non-covariant term to the GR Lagrangian which enforces the harmonic conditions:

$$(71) \quad L = \Xi_\alpha g^{\alpha\alpha} \sqrt{-g} + L_{GR}(g^{\alpha\beta}, \psi^{matter})$$

For some constants Ξ_α . Its dependence on the preferred coordinates $X^\alpha(x)$ can be made explicit ³

$$(72) \quad L = \frac{-1}{2} \Xi_\gamma g^{\alpha\beta} X_{,\alpha}^\gamma X_{,\beta}^\gamma \sqrt{-g} + L_{GR}(g^{\alpha\beta}, \psi^{matter})$$

This explicit form is useful because it allows variation over the preferred coordinates. We have to take care — the four functions $X^\alpha(x) + \delta X^\alpha(x)$ have to define a valid system of coordinates — but nonetheless variation is possible and gives Euler-Lagrange equations for the Ξ^α of the same form as for usual fields. We obtain:

$$(73) \quad \frac{\delta S}{\delta X^\gamma} = \Xi_\gamma \partial_\beta (g^{\alpha\beta} \sqrt{-g} \partial_\alpha X^\gamma),$$

thus, the preferred coordinates X^α are harmonic. The Lagrangian (71) obviously defines a metric theory of gravity with Einstein equivalence principle. In the limit $\Xi_\alpha \rightarrow 0$ we obtain the Einstein equations. The terms $g^{\alpha\alpha} \sqrt{-g}$ do not depend on partial derivatives of the metric, therefore the limit $\Xi_\alpha \rightarrow 0$ is natural for small distances and weak fields.

But, as long as we simply postulate the Lagrangian (72), we have no explanation for these properties. The classical argument against the Lorentz ether may be raised: It needs some conspiracy, does not give an explanation for relativistic symmetry. Is it possible to derive this Lagrangian from some postulates which are more natural for a theory with preferred frame?

Theorem 4. *The Lagrangian (72) follows from the following two conditions:*

$$(74) \quad \frac{\delta S}{\delta X^0} = \Xi_0 (\partial_t \rho + \partial_i (\rho v^i))$$

$$(75) \quad \frac{\delta S}{\delta X^i} = \Xi_i (\partial_t (\rho v^i) + \partial_j (\rho v^i v^j - \sigma^{ij}))$$

Indeed, given (67), the equations (74), (75) are equivalent to (73). The general solution of (73) is defined by a particular solution (given by the first, non-covariant term of (72)) and the general solution of the homogeneous problem

$$(76) \quad \frac{\delta S}{\delta X^\alpha} = 0,$$

³The dependence of some expression on the preferred coordinates $X^\alpha(x)$ is, by definition, explicit, if, after a formal replacement of occurrence of $X^\alpha(x)$ by four scalar fields $Y^\alpha(x)$, the resulting expression is covariant. So, in a^0 a replacement $X^\alpha(x) \rightarrow Y^\alpha(x)$ changes nothing, the resulting expression a^0 is not covariant, thus, the dependence on the preferred coordinates is implicit. Instead, in the form $a^\mu X_{,\mu}^0$, the replacement gives the expression $a^\mu Y_{,\mu}^0$, which is covariant, because Y^0 is considered as a scalar field.

thus, modulo a covariant Lagrangian. The covariance of the Lagrangian is what we take here as the definition of the Lagrangian L_{GR} of general relativity⁴ in (72) \square .

Now, to postulate the equations (74), (75) does not require much conspiracy. Instead, they can be seen as a combination of the Noether theorem with the standard interpretation of continuity and Euler equations, as conservation laws for energy and momentum in condensed matter theories. Indeed, if the Lagrangian, in its explicit form, has a symmetry $X^\alpha \rightarrow X^\alpha + c$, the Euler-Lagrange equation for the preferred coordinates does not depend on the X^α themselves, but only on its partial derivatives. In this case, the Euler-Lagrange equation automatically obtains the form of a conservation law:

$$(77) \quad \frac{\delta S}{\delta X^\alpha} = -\partial_\beta \left(\frac{\partial L}{\partial_\beta X^\alpha} + \dots \right)$$

Thus, the left hand side of (74), (75) define the Noether conservation laws related with translation in time and space. On the right hand side we have the continuity and Euler equations — the conservation laws for energy and momentum in condensed matter theory. To identify left and right hand sides is a very natural postulate for a condensed matter theory.

More details and consequences of this theory of gravity can be found in [21]. Especially, the gauge-breaking term stops (for the correct sign of the constants) the black hole collapse and prevents the big bang singularity. The condensed matter approach to gravity solves many quantization problems of GR quantization: The notorious “problem of time” [14] simply disappears. Together with the black hole collapse the related information loss problem [20] disappears too.

APPENDIX B. THE DIRAC OPERATOR ON THE EXTERIOR BUNDLE

Let’s remember the basic formulas for the Dirac operator in the exterior bundle (see, for example, [19]). The exterior bundle or de Rham complex $\Lambda = \sum_{k=0}^d \Lambda^k$ consists skew-symmetric tensor fields of type $(0, k)$, $0 \leq k \leq d$ which are usually written as differential forms

$$(78) \quad \psi = \psi_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k} \in \Lambda^k$$

The exterior bundle Λ has dimension 2^d in the d -dimensional space. The most important operation on Λ is the external derivative $d : \Lambda^k \rightarrow \Lambda^{k+1}$ defined by

$$(79) \quad (d\psi)_{i_1 \dots i_{k+1}} = \sum_{q=1}^{k+1} \frac{\partial}{\partial x^{i_q}} (-1)^q \psi_{i_1 \dots \hat{i}_q \dots i_{k+1}}$$

where \hat{i}_q denotes that the index i_q has been omitted. It’s main property is $d^2 = 0$. In the presence of a metric, we have also the important \star -operator $\Lambda^k \rightarrow \Lambda^{d-k}$:

$$(80) \quad (\star\psi)_{i_{k+1} \dots i_d} = \frac{1}{k!} \varepsilon_{i_1 \dots i_d} g^{i_1 j_1} \dots g^{i_k j_k} \psi_{j_1 \dots j_k}$$

⁴Note that we use here the most general understanding of general relativity, where the Einstein-Hilbert Lagrangian is only the lowest order term, and higher order terms, or terms with higher order derivatives in the metric, are, in principle, allowed, as long as they are covariant. This understanding is standard for effective field theories — in the large distance limit, only the lowest order terms survive.

with $\star^2 = (-1)^{k(d-k)} \text{sgn}(g)$. This allows to define a global inner product by

$$(81) \quad (\phi, \psi) = \int \phi \wedge (\star\psi) = \int \psi \wedge (\star\phi)$$

It turns out that the adjoint operator of $d^* : \Lambda^k \rightarrow \Lambda^{k-1}$ of d is

$$(82) \quad d^* = (-1)^{kd+d+1} \star d \star$$

Note that the expressions for \star^2 and d^* depend on the order of the form k , which is not nice. But a minor redefinition of the \star operator allows to solve this problem. For the operator

$$(83) \quad * = i^{k(d-k)} \star$$

the resulting expressions no longer depend on k :

$$(84) \quad *^2 = \text{sgn}(g), \quad d^* = (-i)^{d+1} * d *$$

In this general context we can define the Laplace operator as

$$(85) \quad \Delta = dd^* + d^*d.$$

Then, the Dirac operator (as it's square root) can be defined as

$$(86) \quad D = d + d^*,$$

so that $\Delta = D^2$. Indeed, we have $d^2 = 0$ as well as $(d^*)^2 = 0$.

B.1. Discretization of the Dirac operator. The special geometric nature of the exterior bundle allows to define a nice doubler-free discretization of the Dirac equation on a general cell complex. Such a cell complex consists of cells c_i of dimension $k = \dim(c_i)$, which are embeddings of the k -dimensional unit cube I^k into the manifold so that the image of the boundary is part of the image of lower-dimensional cells of the complex, and the image of all cells of the cell complex covers the whole manifold.

On such a cell complex, k -dimensional differential forms are represented on the lattice by their integrals over the k -dimensional cells c_i of the cell complex:

$$(87) \quad \Psi \rightarrow \{\psi_i\}, \psi_i = \int_{c_i} \Psi$$

The external derivative defines in a similar natural way a derivative for functions on the mesh, with the same most important exact property $d^2 = 0$.

For the definition of the \star -operator we need a metric and a dual mesh. A metric $g_{\mu\nu}$ on the manifold defines in a natural way for every cell c_i its area $a_i = a(c_i) > 0$. For a triangulation on Euclidean background, the values a_i depend on each other. But in the general case they may be considered as independent variables, which approximate the metric on the cell complex. In the following we consider them as given and defining the metric.

A dual mesh is a mesh with cells \hat{c}_i so that for each cell c_i of the original mesh with dimension k we have a corresponding "dual cell" \hat{c}_i of dimension $d - k$ which intersects only the cell c_i , in a single point, orthogonally and with positive intersection index. The metric defines the areas \hat{a}_i of the dual cells in a similar way. Now, the lattice Hodge \star -operator may be defined as

$$(88) \quad \star \psi_i = \frac{\hat{a}_i}{a_i} \hat{\psi}_i$$

and maps functions on the mesh to functions on the dual mesh. Note that the dual of the dual mesh has the same cells as the original mesh, but possibly with different orientations. Therefore, for \star^2 we obtain an additional factor $(-1)^{k(d-k)}$ as in the continuous case.

Thus, we can define the exterior derivative as well as the Hodge \star operator on the lattice preserving their algebraic properties $d^2 = 0$, $\star^2 = (-1)^{k(d-k)}$. As a consequence, the remaining part of the theory can be transferred on the lattice too.

It is a general consequence of the geometric character of the continuous Dirac equation as well as its lattice discretization that the lattice discretization does not have doublers. See, for example, [2].

REFERENCES

- [1] R. Arnowitt, S. Deser, C.W. Misner, The dynamics of general relativity, in L. Witten (ed.), *Gravitation: An introduction to current research*, Wiley, NY, 1962
- [2] P.Becher, H.Joos, The Dirac-Kähler Equation and Fermions on the Lattice, *Z. Phys. C - Particles and Fields* 15, 343-365, 1982
- [3] F.A.Berezin, *The method of second quantization*, Academic Press, N.Y., 1966
- [4] K. Bleuer, *Helv. Phys. Acta*, v.23, nr. 5, p.567-586, 1950
- [5] D. Bohm, *Phys.Rev.* 85, 166-193, 1952
- [6] C. Daviau, Dirac equation in the Clifford algebra of space, in V. Dietrich, K. Habetha, G. Jank (eds.), *proc. of "Clifford algebra and their applications in mathematical physics"* Aachen 1996, Kluver/Dordrecht, 67-87, 1998
- [7] P.A.M. Dirac, *Proc. Roy. Soc. A* 114, 767, p. 243-265, 1927
- [8] E. Fermi, *Rev. Md. Phys.*, v.4, nr.1, p.87-132, 1932
- [9] P. Frampton, S. Glashow, chiral color: an alternative to the standard model, *Phys. Lett.* B190, 157, 1987
- [10] H. Georgi, S. Glashow, Unity of all elementary-particle forces, *Physical Review Letters*, 32, 438, 1974
- [11] S. Gupta, *Proc. Phys. Lett.* B 521, 429, 1950
- [12] R. Gupta, Introduction to lattice QCD, hep-lat/9807028
- [13] D. Hestenes, Space-time structure of weak and electromagnetic interactions, *Found. Phys.* v.12, 153-168 (1982)
- [14] C. Isham, Canonical quantum gravity and the problem of time, gr-qc/9210011, 1992
- [15] A. Kadic, D.G.B. Edelen, A gauge theory of dislocations and disclinations, *Lecture Notes in Physics* vol 174, Springer, Berlin 1983
- [16] E. Kähler, *Rendiconti di Matematica* (3-4) 21, 425, 1962
- [17] E. Nelson, Derivation of the Schrödinger equation from Newtonian mechanics, *Phys.Rev.* 150, 1079-1085, 1966
- [18] J.C.Pati, A.Salam, Lepton number as the fourth color, *Phys. Rev. D* 10, 275, 1974
- [19] G. Pete, Morse theory, <http://www.math.u-szeged.hu/~gpete/morse.ps>
- [20] J. Preskill, Do black holes destroy information? hep-th/9209058, 1992
- [21] I. Schmelzer, a generalization of the Lorentz ether to gravity with general-relativistic limit, gr-qc/0205035, 2002
- [22] L. Susskind: *Phys. Rev. D*16, 3031, 1977
- [23] J.A. Wheeler, *Geometrodynamica*, Academic Press, New York 1962.